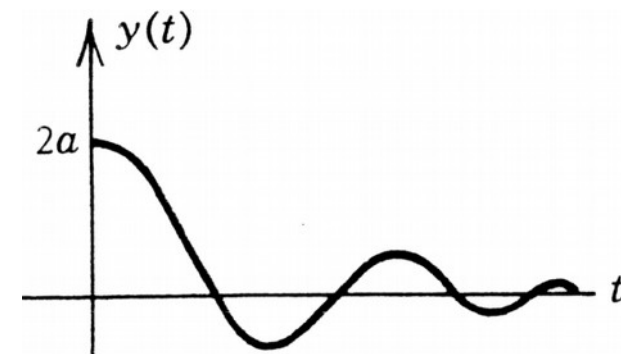


Allgemein

- Kausale Systeme
- Hilbert-Transformation
- Analytische Signale
- Detektion der Einhüllenden
- Minimum und Mixed-Phase Systems
- Modulatoren und Demodulatoren
- Anwendungen

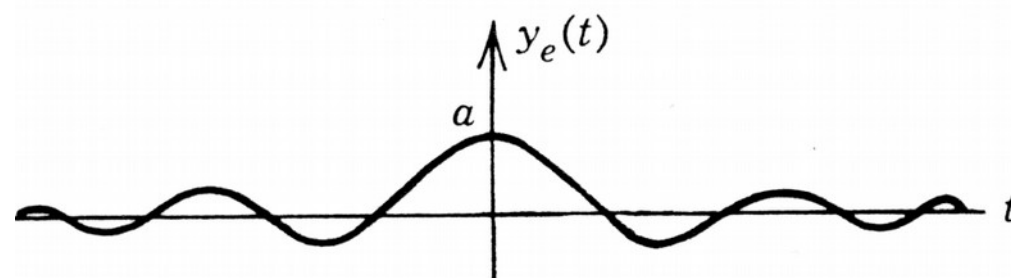
Physikalisch realisierbare Systeme

- Reell: $\Im \{ y(t) \} = 0$
- Stabil: $-M < y(t) < M$
- Kausal: $y(t < 0) = 0$



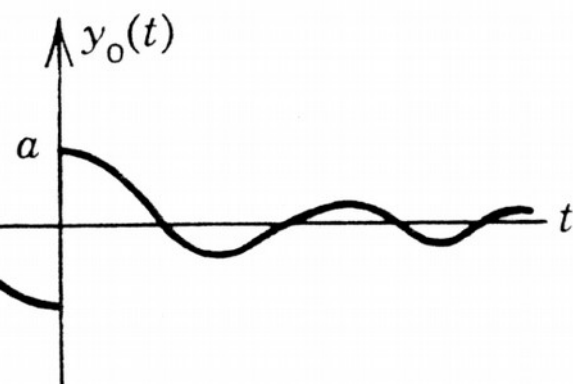
- Zerlegung:

$$y(t) = y_e(t) + y_o(t)$$



- Gerader Anteil:

$$y_e(t) = \frac{1}{2} [y(t) + y(-t)]$$



- Ungerader Anteil:

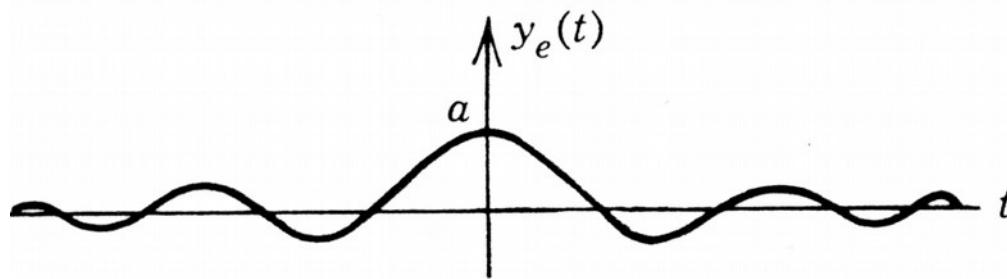
$$y_o(t) = \frac{1}{2} [y(t) - y(-t)]$$

Betrachtung im Frequenzbereich

- $$Y(f) = FT \{ y(t) \} = \int_0^{\infty} y(t) e^{-j2\pi f t} dt = Y_R(f) - j Y_I(f)$$

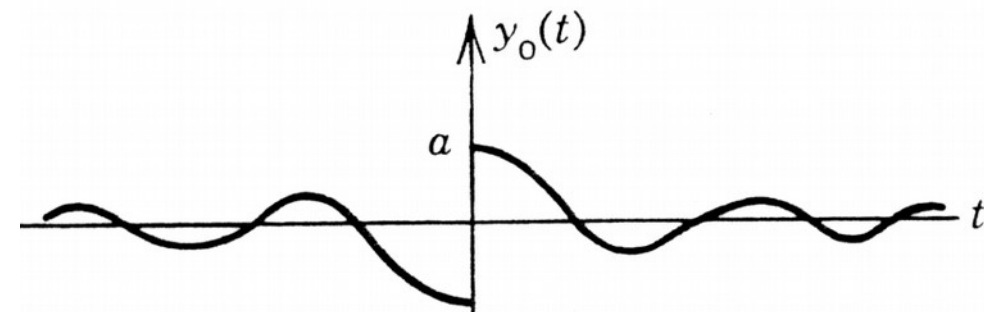
$$Y_R(f) = \int_0^{\infty} y(t) \cos(2\pi f t) dt \quad Y_I(f) = \int_0^{\infty} y(t) \sin(2\pi f t) dt$$

- $$Y(f) = FT \{ y(t) \} = FT \{ y_e(t) + y_o(t) \} = Y_R(f) - j Y_I(f)$$



Spiegelsymmetrisch

$$FT \{ y_e(t) \} = Y_R(f)$$



Schiefsymmetrisch

$$FT \{ y_o(t) \} = -j Y_I(f)$$

Zusammenhänge

$$y_e(t) = \frac{1}{2} [y(t) + y(-t)]$$

$$y_o(t) = \frac{1}{2} [y(t) - y(-t)]$$

$$\text{für } t > 0: \quad y_e(t) = \frac{1}{2} y(t)$$

$$y_o(t) = \frac{1}{2} y(t) \longrightarrow y_e(t)$$

$$\text{für } t < 0: \quad y_e(t) = \frac{1}{2} y(-t)$$

$$y_o(t) = -\frac{1}{2} y(-t) \longrightarrow -y_e(t)$$

$$y_o(t) = \text{sgn}(t) y_e(t)$$

$$FT \{ y_o(t) \} = FT \{ \text{sgn}(t) y_e(t) \} = FT \{ \text{sgn}(t) \} * FT \{ y_e(t) \}$$

$$FT \{ \text{sgn}(t) \} = \frac{-j}{\pi f} : \quad FT \{ y_o(t) \} = \int_{-\infty}^{\infty} \frac{-j FT \{ y_e(t) \}}{\pi (f - u)} du$$

$$FT \{ y_e(t) \} = Y_R(f)$$

$$\longrightarrow Y_I(f) = \int_{-\infty}^{\infty} \frac{Y_R(u)}{\pi (f - u)} du \longrightarrow Y_R(f) * \frac{1}{\pi f}$$

$$FT \{ y_o(t) \} = -j Y_I(f)$$

$$Y_I(f) = HT \{ Y_R(f) \}$$

Hilbert-Transformation

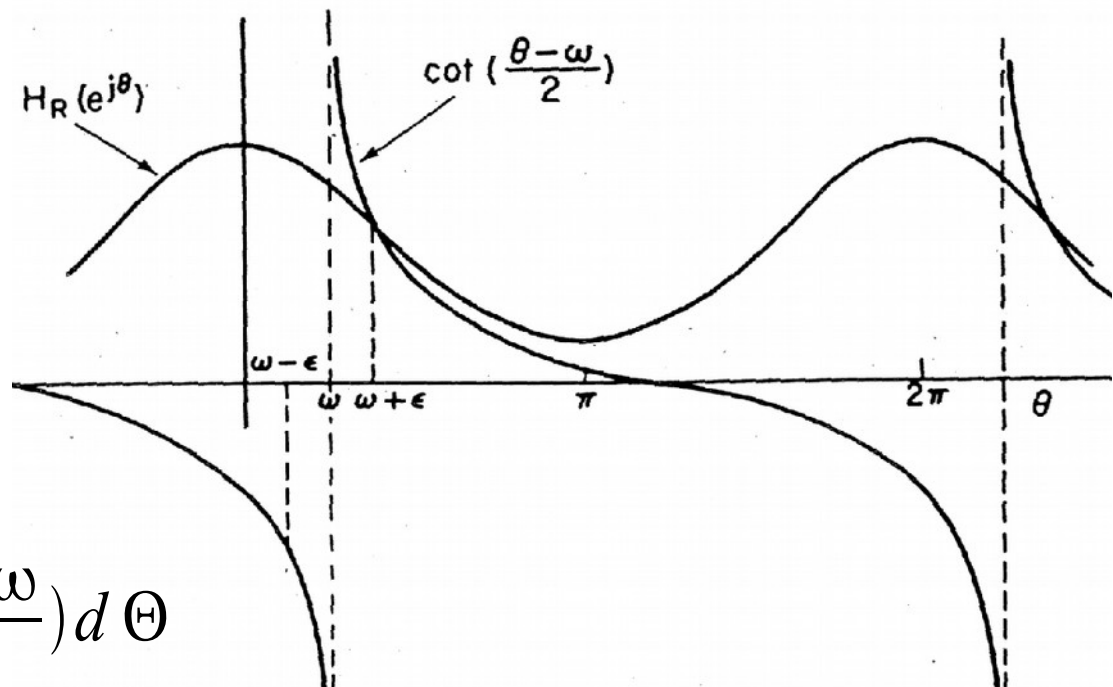
- Faltungsintegral: $\tilde{x}(v) = \left(x * \frac{1}{\pi v} \right) (v)$
- $$\tilde{x}(v) = HT \{ x(v) \} = \int_{-\infty}^{\infty} \frac{x(u)}{\pi \cdot (v-u)} du$$

- Im Frequenzbereich:

$$Y_I(f) = \int_{-\infty}^{\infty} Y_R(u) \cdot \frac{1}{\pi \cdot (f-u)} du$$

diskretisiert

$$Y_I(e^{j\omega}) = \frac{1}{2\pi} P \int_{-\pi}^{\pi} Y_R(e^{j\omega}) \cdot \cot\left(\frac{\Theta - \omega}{2}\right) d\Theta$$



Hilbert-Transformation

- $(\pi/2)$ -Phasenshifter: $\tilde{x}(t) = \left(x * \frac{1}{\pi t} \right) (t)$

$$FT \{ 1/(\pi t) \} = -j \operatorname{sgn}(f)$$

$$\tilde{X}(f) = \underbrace{-j \operatorname{sgn}(f)} X(f)$$

$$= \begin{cases} -j & \text{für } f > 0 \\ 0 & \text{für } f = 0 \\ j & \text{für } f < 0 \end{cases} = \begin{cases} e^{-j(\pi/2)} & f > 0 \\ 0 & f = 0 \\ e^{j(\pi/2)} & f < 0 \end{cases}$$

$$\tilde{X}(f) = |\tilde{X}(f)| \cdot e^{-j\tilde{\phi}_x}$$

$$|\tilde{X}(f)| = |X(f)|$$

$$\tilde{\phi}_x(f) = \phi_x(f) \pm (\pi/2) \quad \begin{cases} + & \text{für } f > 0 \\ - & \text{für } f < 0 \end{cases}$$

Analytisches Signal

- Analytisches Signal:

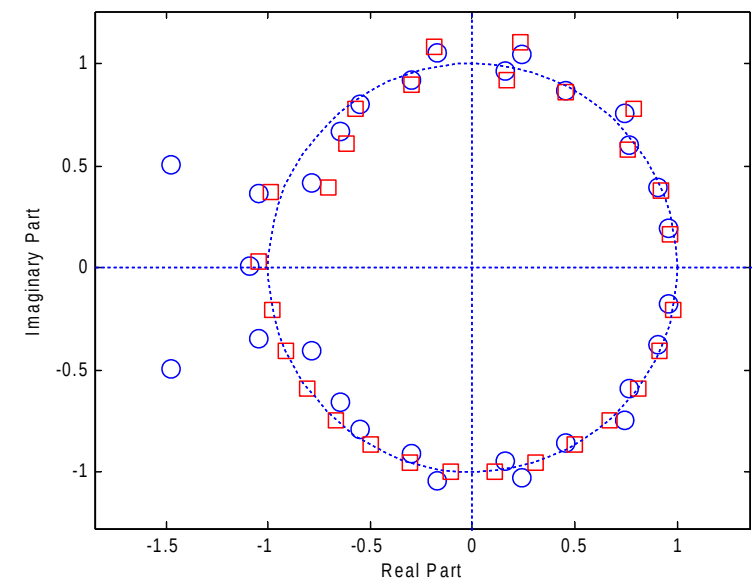
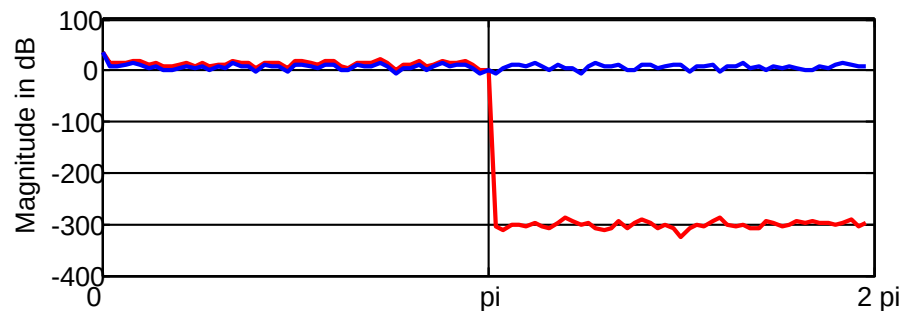
$$z(t) = x(t) + j\tilde{x}(t) \longrightarrow \mathfrak{I}\{z(t)\} = HT[\Re\{z(t)\}]$$

- Kausales Spektrum: $Z(f) = 0 \quad \forall f < 0$

- Diskret:

$$z[n] = x[n] + j\tilde{x}[n]$$

$$Z(e^{j\Theta}) = 0 \quad \forall -\pi < \Theta < 0$$



Hilbert-Transformer

- Kausales Spektrum: $Z(e^{j\Theta})=0 \quad \forall -\pi < \Theta < 0$

$$Z_E(e^{j\Theta}) = 1/2 [Z(e^{j\Theta}) + Z^*(e^{-j\Theta})] \quad jZ_O(e^{j\Theta}) = Z_E(e^{j\Theta}) \text{ für } 0 \leq \Theta \leq \pi$$

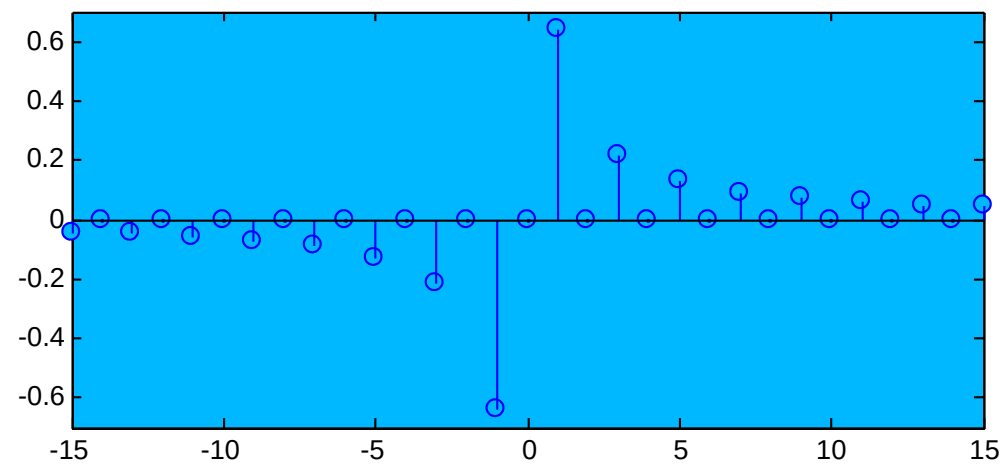
$$jZ_O(e^{j\Theta}) = 1/2 [Z(e^{j\Theta}) - Z^*(e^{-j\Theta})] \quad jZ_O(e^{j\Theta}) = -Z_E(e^{j\Theta}) \text{ für } -\pi \leq \Theta < 0$$

$$Z_O(e^{j\Theta}) = H(e^{j\Theta}) \cdot Z_E(e^{j\Theta}) \quad \text{mit } H(e^{j\Theta}) = \begin{cases} -j & \text{für } 0 \leq \Theta \leq \pi \\ j & \text{für } -\pi \leq \Theta < 0 \end{cases}$$

- Im Zeitbereich:

$$h[n] = \frac{2}{\pi} \frac{\sin^2(\pi n/2)}{n}$$

$$n \neq 0, \quad h[0] = 0$$



Berechnung des analytischen Signals

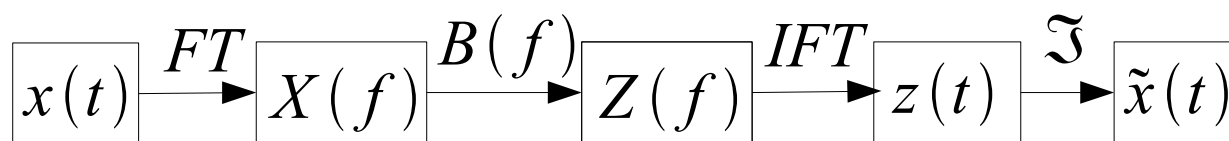
$$\tilde{x}(t) = \left(x * \frac{1}{\pi t} \right) (t) \longrightarrow \tilde{X}(f) = -j \operatorname{sgn}(f) X(f)$$

$$z(t) = x(t) + j \tilde{x}(t)$$

$$Z(f) = X(f) + j \tilde{X}(f)$$

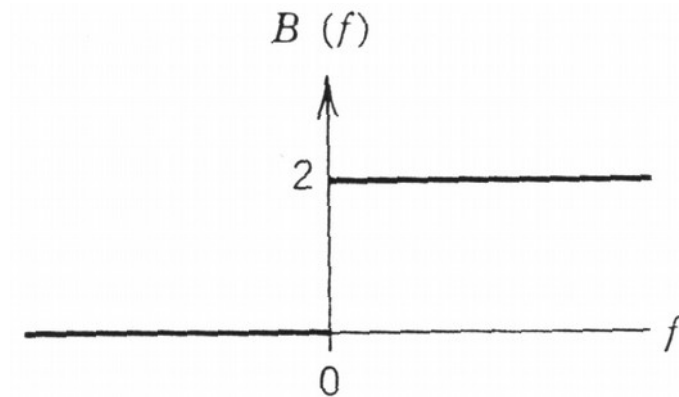
$$Z(f) = [1 + \operatorname{sgn}(f)] \cdot X(f) = B(f) \cdot X(f)$$

$$B(f) = \begin{cases} 2 & f > 0 \\ 1 & f = 0 \\ 0 & f < 0 \end{cases}$$



Diskret:
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$\tilde{x}[n] = 2 \cdot \mathfrak{I} \left\{ \sum_{k=1}^{N/2} X[k] e^{j2\pi kn/N} \right\}$$



Detektion der Einhüllenden

$$z(t) = x(t) + j\tilde{x}(t)$$



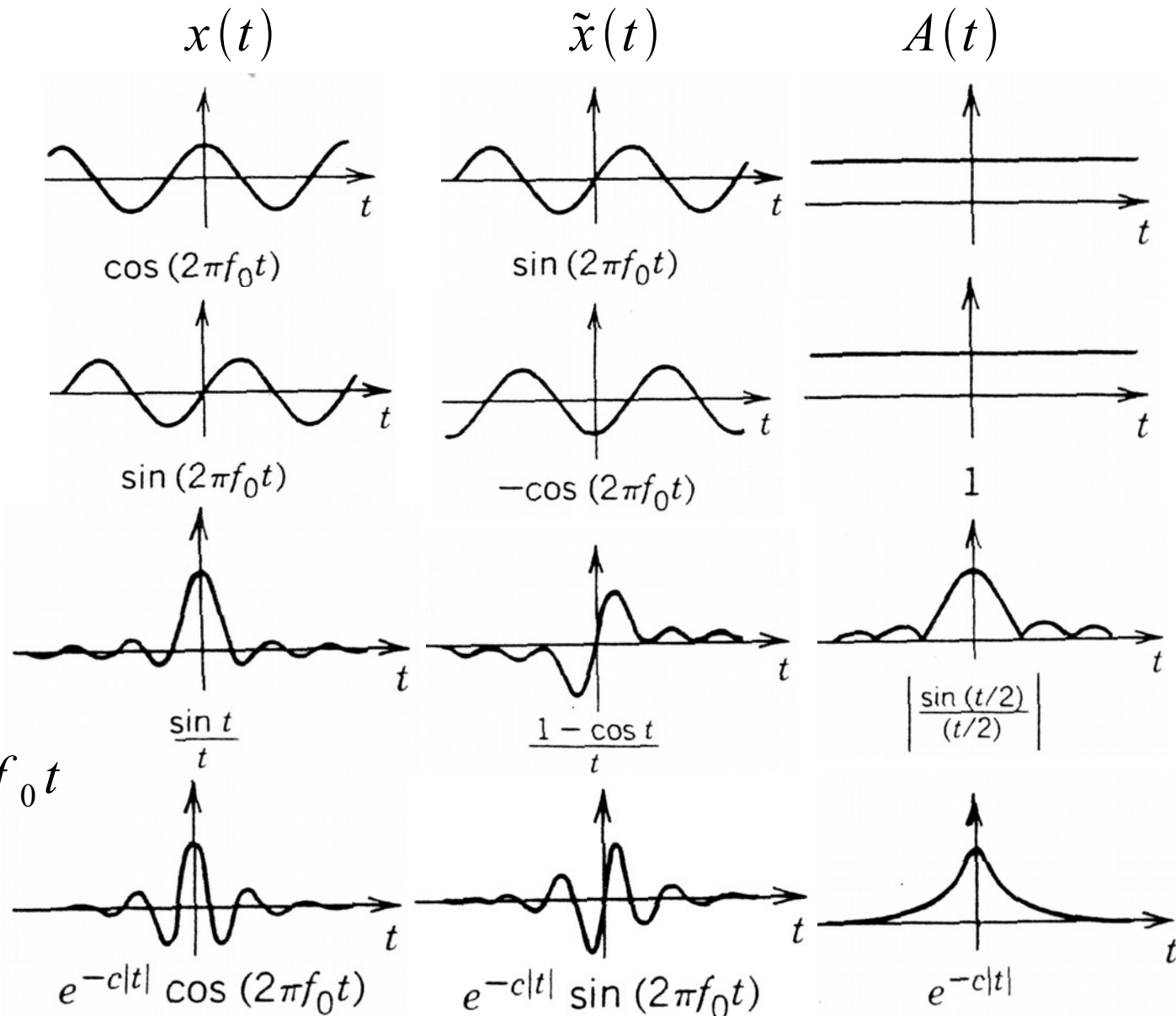
$$z(t) = A(t) \cdot e^{j\phi(t)}$$

Einhüllende:

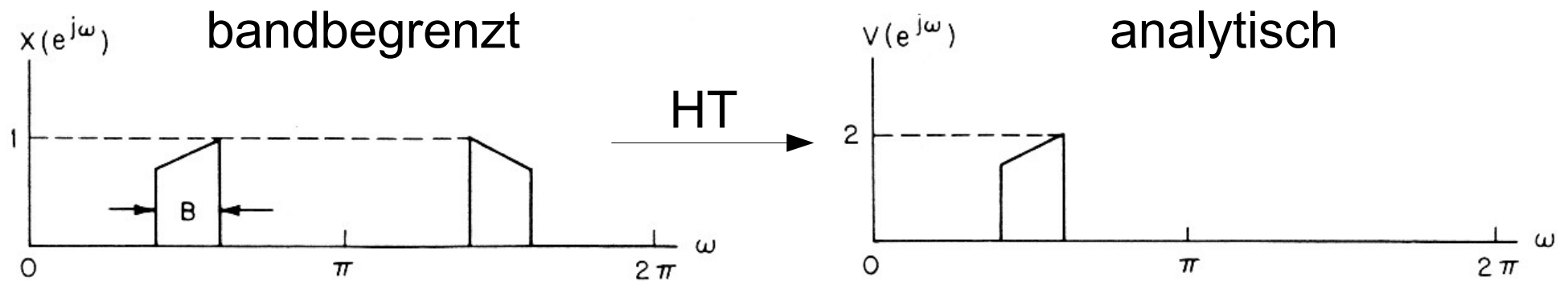
$$A(t) = \sqrt{x^2(t) + \tilde{x}^2(t)}$$

Momentanfrequenz:

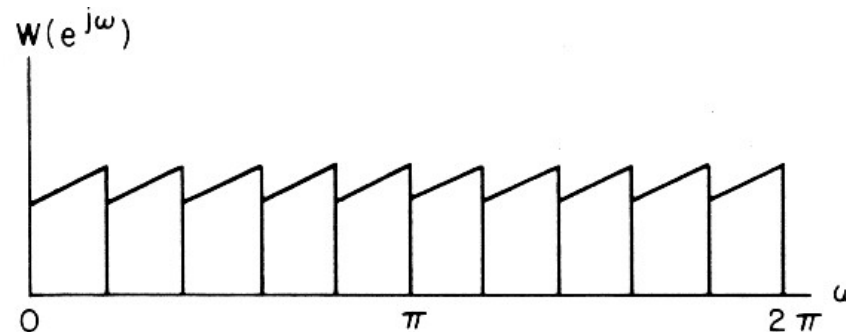
$$\phi(t) = \cot\left(\frac{\tilde{x}(t)}{x(t)}\right) = 2\pi f_0 t$$



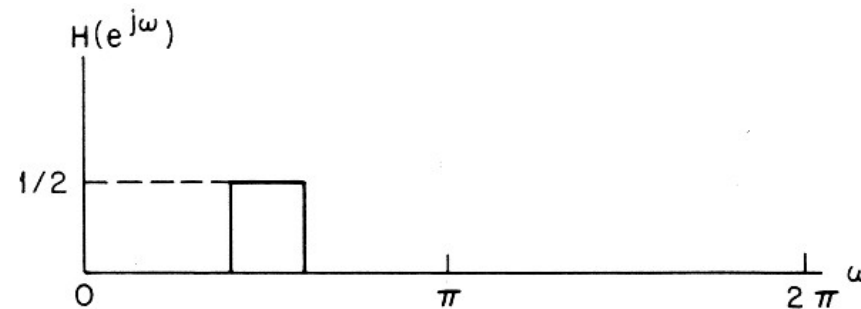
Anwendung: Bandpass Sampling



Unterabtastung:
 $T_S = 1/B$



Rekonstruktionsfilter:



Beziehung: Betrag und Phase

- Reell, kausal, stabil: $h[n] \rightarrow H(f) = |H(f)| e^{j \arg[H(f)]}$

$$\Im \{ H(f) \} = HT [\Re \{ H(f) \}]$$

$$H'(f) = \ln[H(f)] = \ln |H(f)| + j \arg[H(f)]$$

$$\Re \{ H'(f) \} = \ln |H(f)| \qquad \Im \{ H'(f) \} = \arg[H(f)]$$

$$\arg[H(f)] = HT [\ln |H(f)|]$$

- Berechnung der Phase aus dem Betrag:

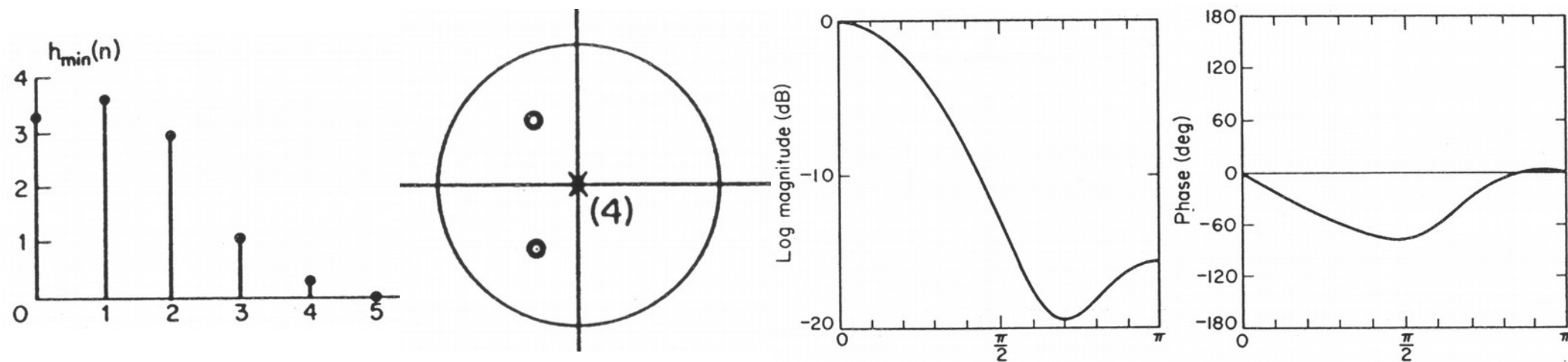
$$\arg[H(e^{j\Theta})] = \frac{1}{2\pi} P \int_{-\pi}^{\pi} \ln |H(e^{j\Theta})| \cot\left(\frac{\delta - \Theta}{2}\right) d\delta$$

Minimum-Phase Systems

- Probleme:
 - Nullstellen außerhalb des EK?
 - $\ln(0) = -\infty$
 - Jede Nullstelle wird zu einer Polstelle
 - Polstellen außerhalb des EK? → System wird instabil!
 - Lösung: Null- und Polstellen innerhalb des EK!
 - Sprünge in der Phase?
 - Phase nur zwischen 0 und 2π definiert
 - Lösung: Phasenentzerren (unwrap)
- Gesamtlösung: Minimalphasige Systeme

Minimum-Phase Systems

- Null- und Polstellen innerhalb des EKs



- Bildung eines inversen Systems:

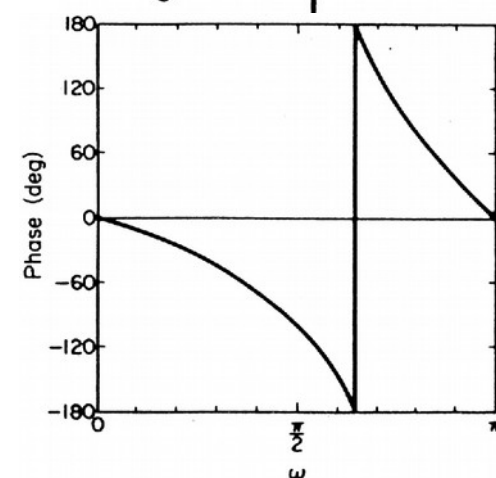
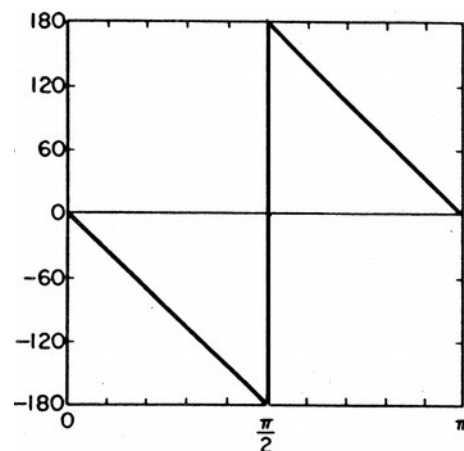
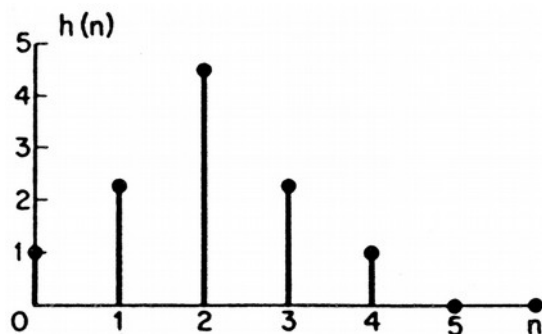
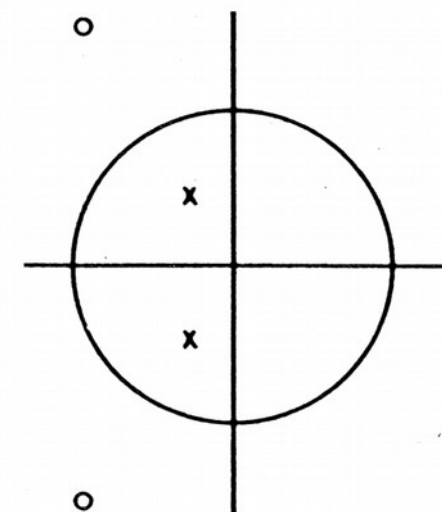
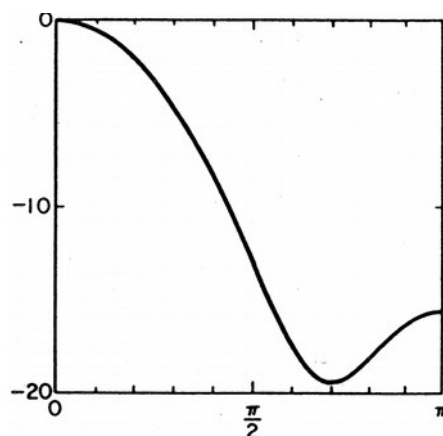
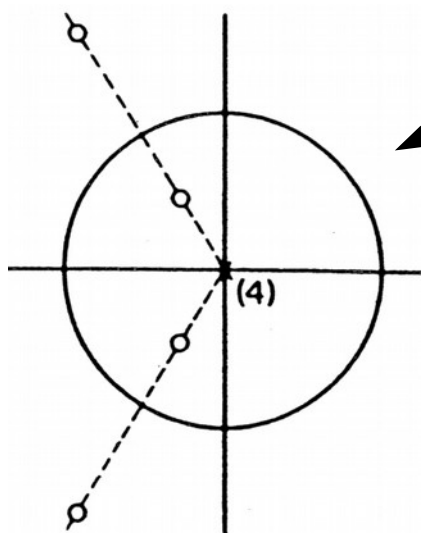
$$H(Z) \cdot H^{-1}(Z) = 1$$

$$H(Z) = \frac{N(Z)}{P(Z)} \longrightarrow H^{-1}(Z) = \frac{P(Z)}{N(Z)}$$

Mixed-Phase Systems

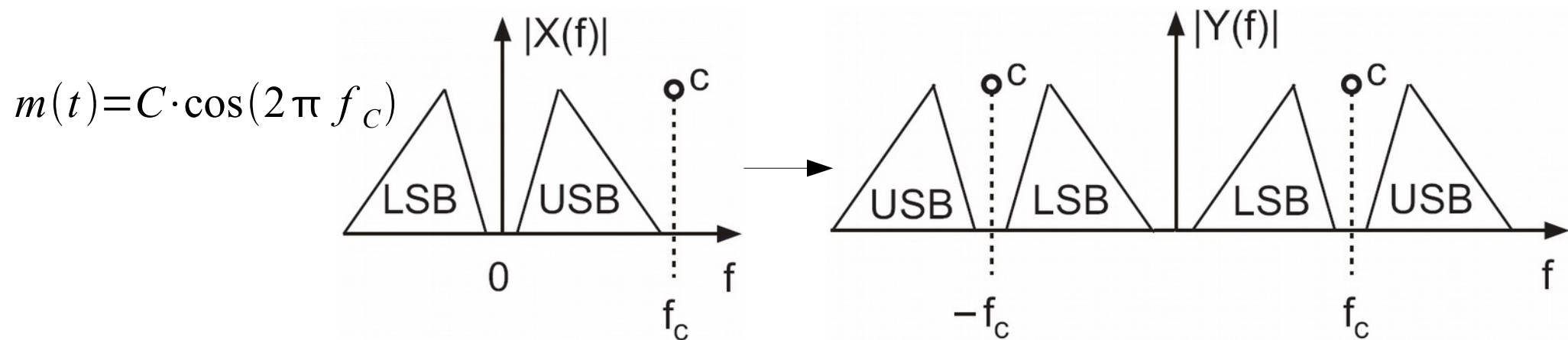
- Mixed Phase = Minimum Phase + Allpass

$$H_{mix}(Z) = H_{min}(Z) \cdot H_{allpass}(Z)$$



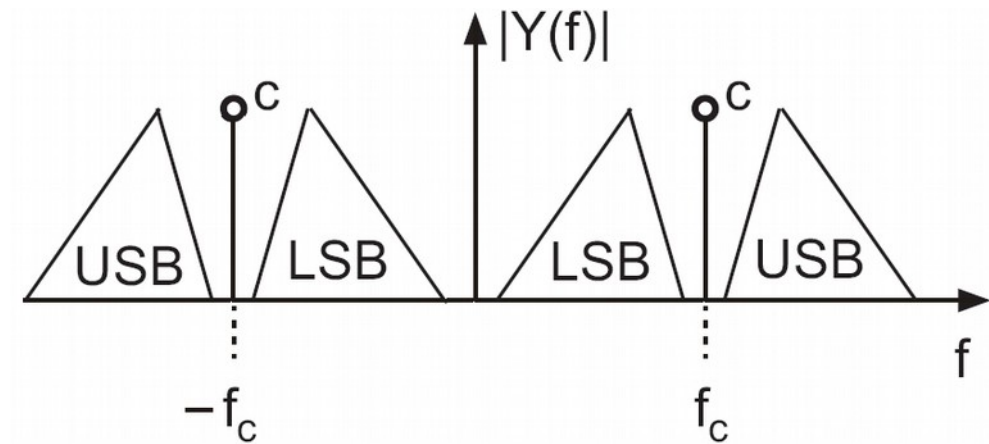
Modulatoren

- Ringmodulation: $y(t) = x(t) \cdot m(t)$



- Amplitudenmodulation:

$$y(t) = [1 + c \cdot x(t)] \cdot m(t)$$



Einseitenbandmodulation

- Bandbegrenztes Signal:

$$x(t)$$

- Analytisches Signal:

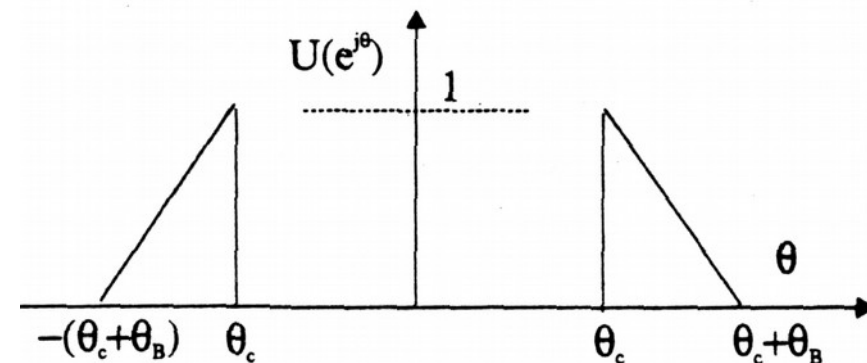
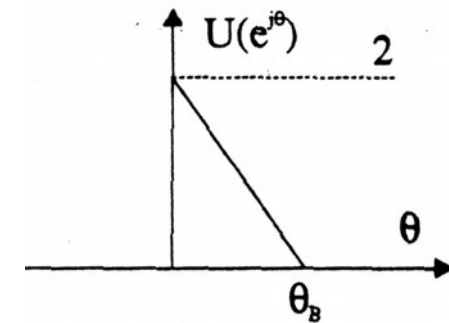
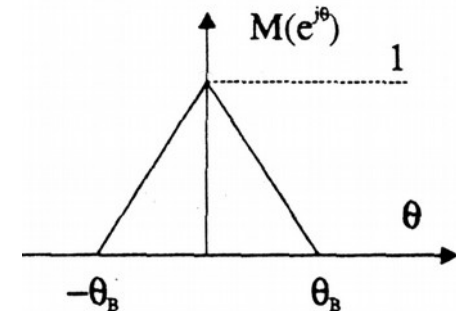
$$z(t) = x(t) + j\tilde{x}(t)$$

- Modulation:

$$y(t) = \Re \left\{ z(t) \cdot e^{j\theta_c t} \right\}$$



$$y(t) = x(t) \cos(\theta_c t) - \tilde{x}(t) \sin(\theta_c t)$$



Einseitenbandmodulation

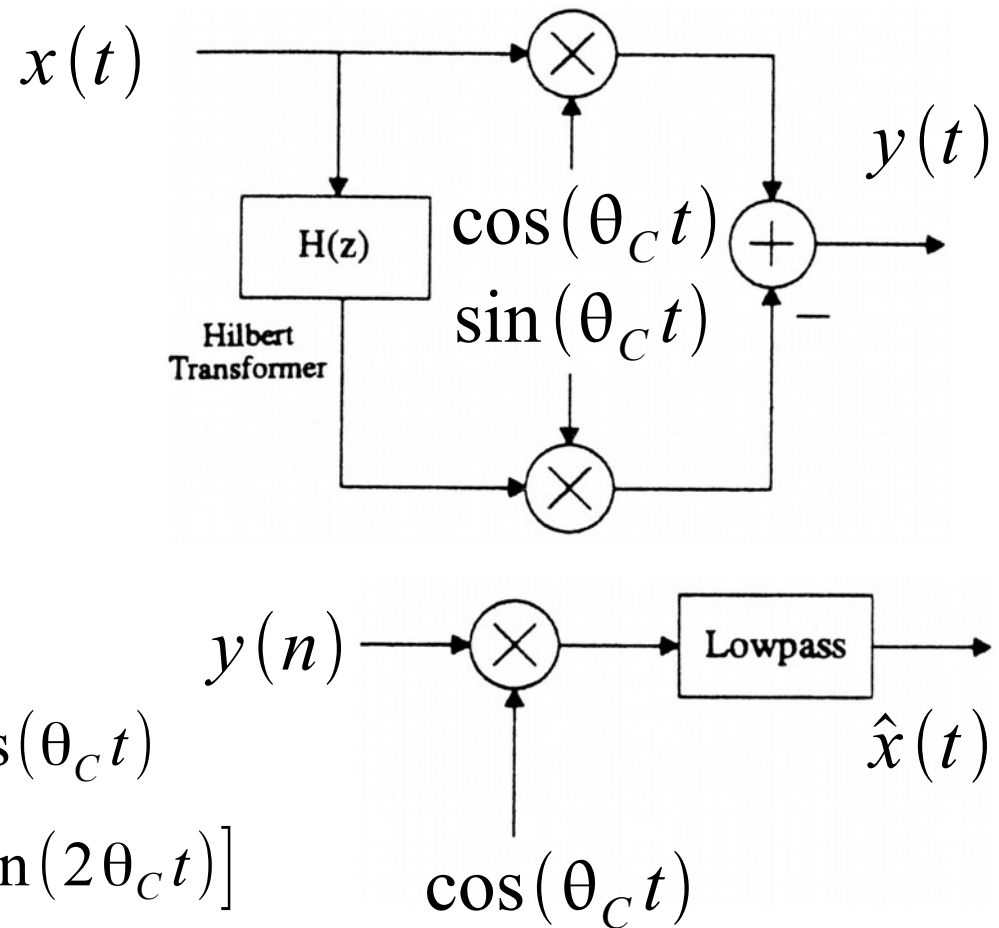
- Modulation:
- Demodulation:
 - Modulation mit Träger:

$$\longrightarrow y(t) \cos(\theta_C t)$$

$$= x(t) \cos^2(\theta_C t) - \tilde{x}(t) \sin(\theta_C t) \cos(\theta_C t)$$

$$= \frac{1}{2} [x(t) + x(t) \cos(2\theta_C t) - \tilde{x}(t) \sin(2\theta_C t)]$$

$$\hat{x}(t) = LP \{ y(t) \cos(\theta_C t) \}, B = \pm \theta_C$$



Demodulatoren

- Detektoren:
 - Halbwellengleichrichtung: $y(n) = |x(n)|$ für $x(n) > 0$
 - Vollwellengleichrichtung: $y(n) = |x(n)|$ für alle $x(n)$
 - Quadrierer: $y(n) = x(n)^2$
 - Envelope Detector: $y(n) = \sqrt{x^2(n) + \tilde{x}^2(n)}$
- Integrierer:
 - Meist einfache Tiefpassfilter + Attack/Release Time
- Skalierer:
 - Skalierung der Größen auf die gewünschte Skala

Verwendung der Demodulatoren

- VU-Meter:
 - AM-Detektor = RMS + Tiefpaß
- Peak Program Meter:
 - Vollwellengleichrichter + AR-Averager
- RMS-Anzeige:
 - Quadrierer + auf Wurzelfunktion skalierte Anzeige
- Vocoder, Live-Electronics:
 - RMS-Detektor = Quadrierer + AR-Averager
- Envelope Detector: Differenztonbildung

Differenztonbildung

- **Signal:** $x(t) = A \cos(2\pi f_1 t) + B \cos(2\pi f_2 t)$
- **Ziel:** $y(t) = \cos[2\pi t(f_1 - f_2)]$

$$z(t) = x(t) + j\tilde{x}(t)$$

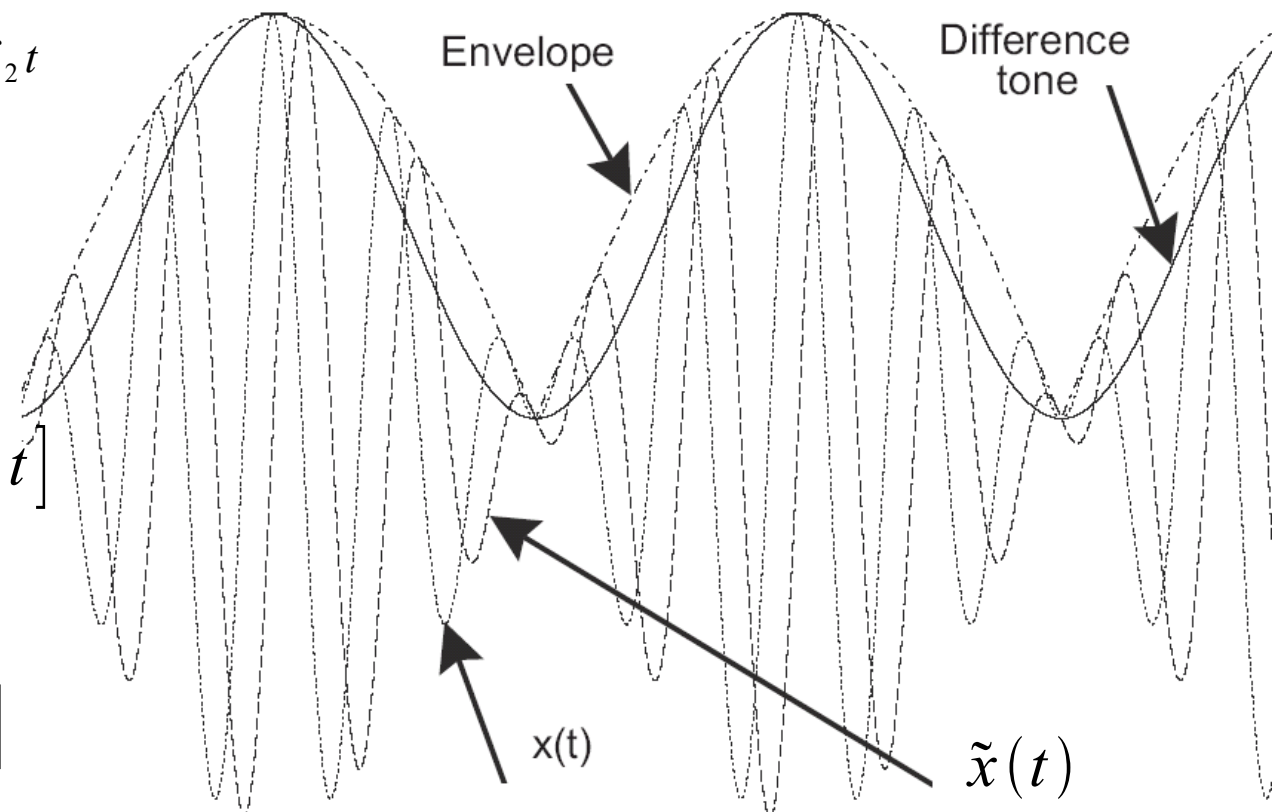
$$z(t) = A e^{j2\pi f_1 t} + B e^{j2\pi f_2 t}$$

$$y(t) = x^2(t) - \frac{1}{2} \Re\{z^2(t)\}$$

$$\frac{A^2 + B^2}{2} + AB \cos[2\pi(f_1 - f_2)t]$$

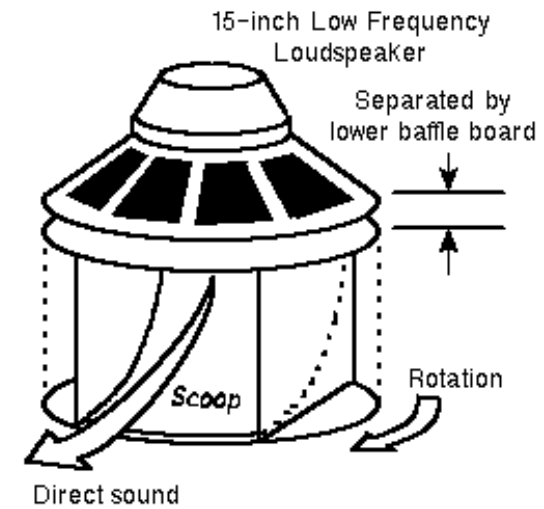
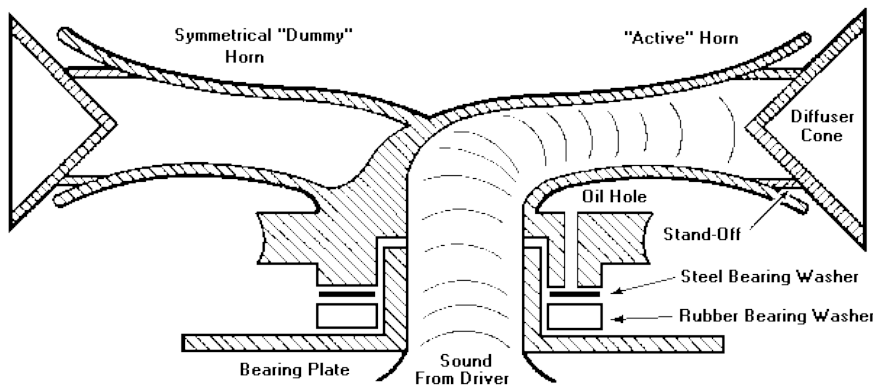
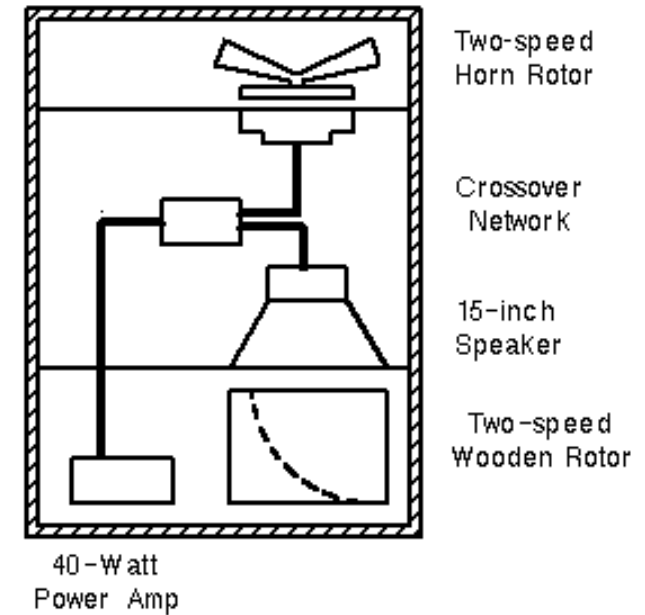
nach einer DC-Sperre:

$$\rightarrow y(t) = \cos[2\pi(f_1 - f_2)t]$$



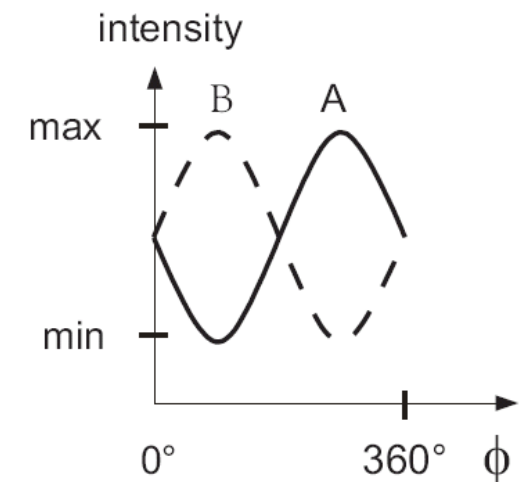
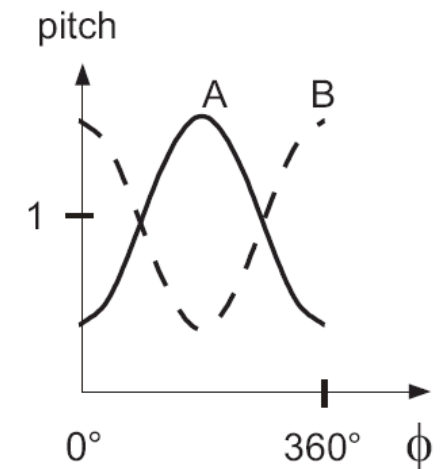
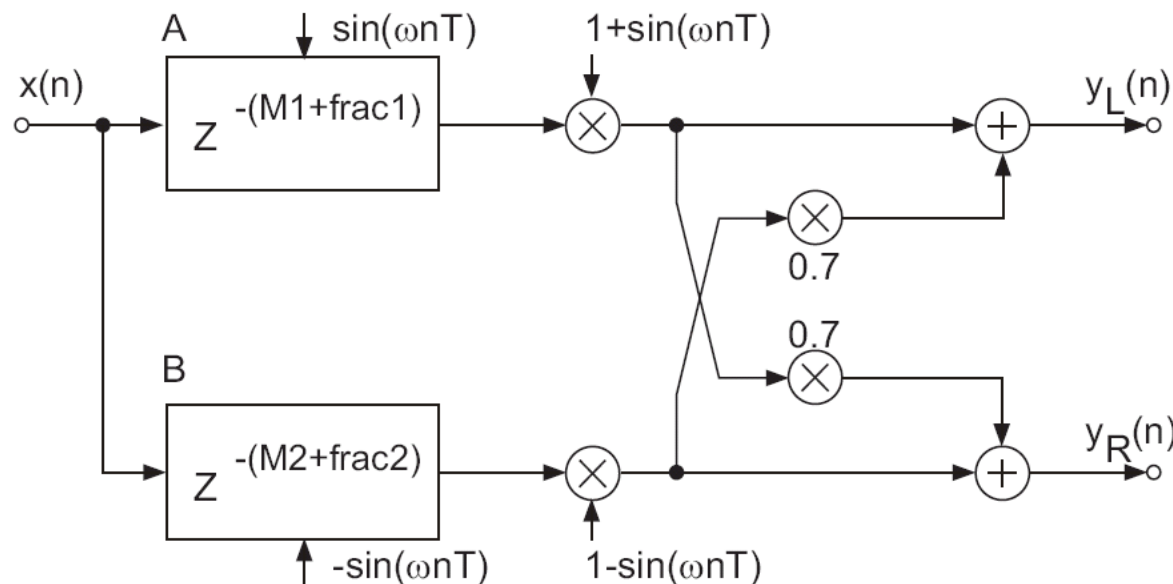
Rotary Speaker Effect

- Rotierende Lautsprecher:
 - Hammond-Orgel
 - Leslie-Effekt



Rotary Speaker Effect

- Simulation:
 - Pitch-Modulation (Dopplereffekt)
 - Amplituden-Modulation
 - Stereo-Effekt durch Verstimmung



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